Israel conditions for the Gauss-Bonnet theory and the Friedmann equation on the brane universe

Elias Gravanis, Steven Willison

Department of Physics, King's College London,
Strand, London WC2R 2LS, United Kingdom.

Abstract

Assuming an Einstein-Gauss-Bonnet theory of gravitation in a ($D \geq 5$)-dimensional spacetime with boundary, we consider the problem of the boundary dynamics given the matter Lagrangian on it. The resulting equation is applied in particular on the derivation of the Friedmann equation of a 3-brane, understood as the non-orientable boundary of a 5d spacetime. We briefly discuss the contradictory conclusions of the literature.

In the framework of Einstein gravitational theory, the cosmology of the brane-world model has been first studied in [1]. There the domain-wall/brane is described as a 4d world- volume slice of a 5d spacetime, which is of constant spatial curvature. In [2] it was shown that these solutions and the ones in [3] describe the same physical system which consists in a 3-surface of Robertson-Walker world-volume moving in the background of a 5d AdS-Schwarzchild black hole [1]. Inverting this course one obtains a procedure for constructing brane cosmological solutions that can be applied to other gravitational theories. Different matter content on the 3-brane corresponds to a different 4d trajectory in the bulk black hole spacetime. The dynamics of the 4d trajectory is given by the appropriate Friedmann equation, which is obtained either by matching the bulk solutions around the brane or by viewing the 3-wall as boundary of the spacetime(s) in the spirit of [4], as explained nicely in [5].

In order to extend these results in the case where the gravitational theory is corrected by higher order terms in the curvature, it is at least convenient to study the case where we add to the Einstein theory the Gauss-Bonnet term. This particular combination of the Ricci scalar, tensor and Riemann tensor which reads $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, leads to second order field equations which is not true for a general combination of these tensors. Its special properties in $D(\geq 5)$ dimensions are not unrelated to its nature as a topological invariant in 4d [6] or its origin from strings [7] [2].

The black hole solution studied in [8] is the general solution of the Einstein-Gauss-Bonnet theory in 5d spacetime with constant spatial curvature, that respects Birkhoff's theorem [9]. We would like to derive the equation of motion of a 3-surface with constant spatial curvature, that determines its 4d trajectory in the 5d bulk or, in other words the Friedmann eq. of its Robertson-Walker world-volume, given the energy momentum tensor T^μ_ν of the surface.

Put in general terms, the actual problem one is invited to solve is to find the appropriate junction conditions around a surface filled with matter existing in the spacetime, where a certain given gravitational theory determines its metric. As mentioned above, a way to do this is by treating each side of the surface as boundary of the respective part of the spacetime. Such a method is theoretically appealing because the surface truly separates two smooth manifolds. Also, the quantities involved in the junction conditions are fundamental from that point of view, thus naturally arising in the expressions.

We study then the following problem: there is a $D(\geq 5)$-spacetime M with a boundary ∂M. In the interior of such a spacetime, $M - \partial M$, the "bulk", we want the metric to be determined by the field equations given by the minimization of the action of the Einstein-Gauss-Bonnet theory.

[1] More precisely, the construction is the following: the 3-surface is the boundary of "half" such a space. Its location with respect to the black hole is determined by the boundary dynamics. One introduces a mirror image half, with respect to the boundary, imposing also a $\mathbf{Z_2}$ symmetry to avoid overcounting. The 3-surface becomes the non-orientable boundary of the new space. The evolution/motion of the brane just recovers parts of the bulk spacetime which have been cut off. This is because they are both of constant 3-curvature.

[2] See also the discussion in [10].

We assume that in the boundary there is localized a non-zero energy momentum tensor. Now, in general, varying the action in the presence of the boundary, one is left with normal derivatives of the variation of the metric on the boundary which cannot go away by a partial integration. One should then add a term in the action, local on the boundary, which contributes similar terms that cancel the previous ones and one is left with a well defined variational principle. The needed boundary terms for the Einstein and Gauss-Bonnet theories were derived quite long ago and they belong to the general class of Chern-Simons boundary terms (see references below). Varying the full bulk plus boundary action with respect to the induced metric on the boundary one obtains the equation of motion for it, given in the text by equation (12). Note that no assumptions have been made and such an equation describes the motion of a general boundary given the energy momentum tensor on it, in a bulk which can contain matter as well. Combining then two manifolds like M and identifying the boundaries one obtains the result for the surface mentioned at the beginning of this paragraph, given in eq.(13). This is the junction condition for an arbitrary surface in any bulk solution of the Einstein-Gauss-Bonnet gravity in a $D(\geq 5)$-dimensional spacetime, which offers a more general approach than previous work. If we impose a $\mathbf{Z}_2$ symmetry identifying symmetrical points around the surface we obtain a still general result, eq.(14). This equation looks like (12) but the assumed manifold is qualitatively different; we comment on that in the text. As an application of the junction conditions we treat the specific problem mentioned above, deriving the Friedmann equation/equation of motion of a "brane universe" assuming Einstein-Gauss-Bonnet gravity in the bulk, a problem already solved in [9], resulting in (20).

Assume then that the gravity is described by Einstein-Gauss-Bonnet theory in the bulk of a $(D \geq 5)$-dimensional spacetime which is a manifold M with boundary ∂M. The theory takes an elegant form and calculations are easier if we write everything in terms of differential forms. We use the notation of [10]. Let E^A be the normalized basis of 1-forms in terms of which the metric is $g = \eta_{AB} E^A \otimes E^B$ with $\eta_{AB} = (- + + + ..)$. Also define

$$e_{A_1..A_m} = \frac{1}{(D-m)!}\,\epsilon_{A_1..A_m A_{m+1}..A_D} E^{A_{m+1}} \wedge .. \wedge E^{A_D} \tag{1}$$

where $\epsilon_{A_1..A_D}$ is the completely antisymmetric tensor with the normalization $\epsilon_{0..D-1} = 1$. The curvature 2-form Ω^{AB} in terms of the connection ω^{AB} and the Riemann tensor $R^A_{\ BCD}$ is $\Omega^A_{\ B} = d\omega^A_{\ B} + \omega^A_{\ C} \wedge \omega^C_{\ B} = \frac{1}{2} R^A_{\ BCD} E^C \wedge E^D$. The second fundamental form θ^{AB} is defined as follows: if we introduce Gaussian normal coordinates (x, w) chosen so that the (time-like) boundary ∂M is given by $w = 0$, the metric is written as $ds^2 = dw^2 + \gamma_{\mu\nu}(x, w) dx^\mu dx^\nu$. We define ω_0 as the connection of the product metric that agrees with the previous metric at ∂M: $ds^2 = dw^2 + \gamma_{\mu\nu}(x, 0) dx^\mu dx^\nu$. Clearly this connection has non-zero components only tangentially on the boundary. Then

$$\theta = \omega - \omega_0 \tag{2}$$

or in terms of the extrinsic curvature K^{AB}

$$\theta^{AB} = \theta^{AB}_{\ \ \ C} E^C = (N^A K^B_C - N^B K^A_C) E^C = 2 N^{[A} K^{B]}_C E^C \tag{3}$$

N^A is the normal vector on the boundary. Explicitly the extrinsic curvature reads $K^A_B = -(\eta^{AC} - N^A N^C) \nabla_C N_B$ or in (x, w) coordinates $K_{\mu\nu} = -\frac{1}{2}\ \partial\gamma_{\mu\nu}/\partial w$. Note that θ^{AB} has only mixed (normal - tangential on the boundary) components due to its antisymmetry and the property $N^A K_{AB} = N^B K_{AB} = 0$.

Then the action, discussed explicitly in [10] (see also [11]) , that contains appropriate boundary terms so that the normal derivatives of the variations of the metric cancel identically, can be written

$$\begin{aligned} S = \int_M -2\Lambda\, e + \Omega^{AB} \wedge e_{AB} + \alpha\, \Omega^{AB} \wedge \Omega^{CD} \wedge e_{ABCD} - \\ - \int_{\partial M} \theta^{AB} \wedge e_{AB} + \alpha\, 2\theta^{AB} \wedge (\Omega^{CD} - \frac{2}{3} \theta^C_E \wedge \theta^{ED}) \wedge e_{ABCD} \end{aligned} \tag{4}$$

where we have also introduced a bulk cosmological constant Λ. In the bulk part of the action the linear term in the curvature 2-form is the Einstein-Hilbert action and the quadratic is the Gauss-Bonnet term. We designate α the coupling of the Gauss-Bonnet term which has units (length)^2. If the higher curvature term is thought of as originating from string theory α is proportional to the Regge slope α'. The boundary part is the Chern-Simons term mentioned in the introductory part of our paper. [3]

Varying with respect to the basis forms while keeping the connection fixed we have

$$\delta_E S = \int_M \delta E^F \wedge (-2\Lambda\, e_F + \Omega^{AB} \wedge e_{ABF} + \alpha\, \Omega^{AB} \wedge \Omega^{CD} \wedge e_{ABCDF}) + \tag{5}$$
$$+ \int_{\partial M} \delta E^F \wedge (\theta^{AB} \wedge e_{ABF} + \alpha\, 2\theta^{AB} \wedge (\Omega^{CD} - \frac{2}{3}(\theta \wedge \theta)^{CD}) \wedge e_{ABCDF})$$

If there is no matter in the bulk, the bulk volume integral vanishes giving the field equation outside the boundary. Assuming that there is matter on the boundary, the boundary integral above equals the energy momentum tensor of the matter on it coming from the variation of the matter Lagrangian.

In order to write this in an explicit form in terms of the extrinsic curvature and the intrinsic curvature tensors of the boundary, we first note that by $\omega = \omega_0 + \theta$ we have

$$\Omega^{AB} = \Omega_0^{AB} + d\theta^{AB} + (\omega_0 \wedge \theta)^{AB} + (\theta \wedge \omega_0)^{AB} + (\theta \wedge \theta)^{AB} \tag{6}$$

where $\Omega_0 = d\omega_0 + \omega_0 \wedge \omega_0$ is the intrinsic curvature 2-form of the boundary. Using that in (5) we see that the boundary terms can be written as

$$\int_{\partial M} \delta E^F \wedge (\theta^{AB} \wedge e_{ABF} + \alpha\, 2\theta^{AB} \wedge (\Omega_0^{CD} + \frac{1}{3}(\theta \wedge \theta)^{CD}) \wedge e_{ABCDF}) \tag{7}$$

as the other terms have to have an index on the normal direction that contributes zero as there is a factor θ^{AB} already in the expression. Using the second of the identities

$$E^C \wedge e_{ABF} = \frac{3!}{2!}\delta^C_{[F} e_{AB]}, \quad E^G \wedge E^H \wedge E^I \wedge e_{ABCDF} = \frac{5!}{2!}\delta^I_{[F}\delta^H_D \delta^G_C e_{AB]} = \frac{5!}{2!}\delta^G_{[A}\delta^H_B \delta^I_C e_{DF]}, \tag{8}$$

the first Gauss-Bonnet boundary term gives

$$2\alpha\, \theta^{AB} \wedge \Omega_0^{CD} \wedge e_{ABCDF} = \alpha\, \theta^{AB}_{G}\ R_0^{CD}{}_{HI}\ \frac{5!}{2!}\ \delta^G_{[A}\delta^H_B\delta^I_C e_{DF]} = \alpha\, N^A 5! K^B_{[A}\ R_0^{CD}{}_{BC} e_{DF]} \tag{9}$$

Given that the first three indices are orthogonal to N^A we can write [4]

$$\alpha\, N^A 5! K^B_{[A}\ R_0^{CD}{}_{BC} e_{DF]} = \alpha\, N^A\, 3!\, 2 (K^B_{[B} R_0^{CD}{}_{CD]} e_{FA} - K^B_{[F} R_0^{CD}{}_{BC]} e_{DA} + \tag{10}$$
$$+ K^B_{[D} R_0^{CD}{}_{FB]} e_{CA} - K^B_{[C} R_0^{CD}{}_{DF]} e_{BA})$$

which is easy to see performing the contractions that it takes the form

$$-4\alpha\, N^A\, e_{AB}([4(KR_0)^B_F + 2K^D_C\ R_0^{CB}{}_{DF} - 2KR^B_{0\,F} - K^B_F R_0] + \delta^B_F[KR_0 - 2Tr(KR_0)]) \tag{11}$$

Doing the same for the $\theta \wedge \theta$ term, using $(\theta \wedge \theta)^{CD} = -K^C_H K^D_I\ E^H \wedge E^I$, we can finally write the integrand of (7) in the form

$$\delta E^F \wedge N^A e_{AB}\ [2(K^B_F - \delta^B_F K) + 4\,\alpha(Q^B_F - \frac{1}{3}\,\delta^B_F Q^C_C)], \tag{12}$$
$$Q^B_F = -2K^D_C R_0^{CB}{}_{DF} - 4(R_0 K)^B_F + 2K R^B_{0F} + R_0 K^B_F +$$
$$+ K^B_F(TrK^2 - K^2) + 2K(K^2)^B_F - 2(K^3)^B_F$$

The dynamics of the each boundary is described by setting the quantity in square brackets equal to $-2T^B_F$, for N^A oriented outwards the space according to Stokes theorem and T^B_F is boundary matter energy momentum tensor. The normalization is fixed by the the bulk part of the (5) which reads $-2(\delta^B_F \Lambda + G^B_F + ..)$, where G^B_F is the Einstein tensor and we use a convention such that the field equations have the form $G^B_F + .. = T^B_F$.

[3]We are mainly interested in Chern class type actions in view of their desirable properties. For a construction of the boundary term for a bulk Lagrangian formed as a general polynomial of the Riemann tensor see [12].

[4]Our convention is $R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} + ..$.

With this result at hand we can now treat the problem of a surface in the spacetime in such a theory. We can think of the spacetime as two manifolds $M_i, i = 1, 2$ whose boundaries are identified. The normal vector of the common boundary with respect to each $M_i, i = 1, 2$ are equal up to a sign, $N_1^A = -N_2^A$. The extrinsic curvatures, describing the surface in different spacetimes, will in general differ. The variation of the gravity action with respect to the induced metric is going to give contributions from both sides of the common boundary leading to the following equations of motion for the surface

$$(K_F^B - \delta_F^B K)_1 + 2\,\alpha(Q_F^B - \frac{1}{3}\,\delta_F^B Q_C^C)_1 + (K_F^B - \delta_F^B K)_2 + 2\,\alpha(Q_F^B - \frac{1}{3}\,\delta_F^B Q_C^C)_2 = T_F^B \qquad (13)$$

The suffixes denote that the quantities should be calculated in the indicated side. Note that the intrinsic in the boundary curvature tensors in the definition of Q_B^A do not depend on the side. Note that as it is natural the normal vectors are taken to be oriented inwards the respective M_i.

A particularly interesting restriction to the above is when we identify points of M_1 with points of M_2 around the surface. At each point in the surface, taken to be the origin of the normal direction w we identify symmetrical points. The normal vector at the origin has to have both directions and the origin can be understood as a non-orientable boundary of the spacetime. Practically one, integrating over the boundary, should integrate over both directions. The point is that due to the $\mathbf{Z}_2$ symmetry, both the normal vector and the extrinsic curvature change sign going from one "side" of the boundary to the other, leading to a total factor of 2 in the final result. Also, as it is natural to take the normal vector oriented inwards each half-space, there is an additional minus sign. Then the equation of boundary motion or junction condition under the $\mathbf{Z}_2$ symmetry restriction takes the form

$$2(K_F^B - \delta_F^B K) + 4\,\alpha(Q_F^B - \frac{1}{3}\,\delta_F^B Q_C^C) = T_F^B \qquad (14)$$

We have considered three cases: we can simply have a spacetime with a free boundary; the common boundary described above; and the specific case of a non-orientable boundary with $\mathbf{Z_2}$ symmetry. The qualitative difference between these three case leads to a distinction between the general kind of theories to which they can be applied. In the first case one demands, roughly, that the generic metric function is defined on the half-line, $g(w), w \geq 0$, in the third that its continuation beyond the origin is given by $g(-w) = g(w)$, while in the second no particular restriction is imposed. With $\mathbf{Z_2}$ symmetry, the second derivative in any expression is going to produce a delta function at the origin where the non-orientable boundary is located. This means that there only first derivatives should arise on the boundary terms of the theory which describes the gravitation in the bulk, that is the theory should have second order field equations. As noted earlier, Gauss-Bonnet is the only combination with this property, in the kind of theories which are built by the invariant squares of the curvature tensors, so it is the only one that applies under the $\mathbf{Z_2}$ symmetry. In the other cases a more general combination, whose boundary term can be constructed, should apply.

Let us now apply these results to the case of a 3-wall of constant spatial curvature in the background of the particular black hole solution of the Einstein-Gauss-Bonnet 5d spacetime of constant spatial curvature mentioned earlier. Its line element can be written in the form

$$\begin{aligned} ds^2 &= -f(y)\,dt^2 + \frac{dy^2}{f(y)} + y^2 dx^2, \qquad (15) \\ dx^2 &= \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ f(y) &= k + \frac{y^2}{4\alpha}\left(1 \pm \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{8\alpha\mu}{y^4}}\right) \end{aligned}$$

where μ is the gravitational mass of the black hole. Λ is positive for de Sitter space and should be bounded as $\Lambda \geq -3/4\alpha$. For the sake of the result as well as for its usefulness, we use the method of [2] to transform from the black hole coordinates to the ones built around the trajectory of the wall, where the coordinate w is the proper-"time" of the spacelike geodesics that cross the trajectory vertically. Then one finds that the metric (15) can be also written in another familiar form that reads

$$ds^2 = -\frac{\psi^2(\tau,w)}{\varphi(\tau,w)}d\tau^2 + a^2(\tau)\varphi(\tau,w)dx^2 + dw^2 \tag{16}$$

where the function φ is given implicitly by the equation

$$\int_1^{\varphi} \frac{dx}{V^{1/2}} = \pm 2w \tag{17}$$

where V is given by

$$V = x\left(H^2 + \frac{f(a\sqrt{x})}{a^2}\right) = x\left(H^2 + \frac{k}{a^2}\right) + \frac{x^2}{4\alpha}\left(1 \pm \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{8\alpha\mu}{a^4x^2}}\right) \tag{18}$$

The Hubble parameter is $H = \dot{a}(\tau)/a(\tau)$ and $\psi(\tau,w) = \varphi(\tau,w) + \frac{1}{2H}(\frac{\partial\varphi}{\partial\tau})_w$, where the derivative is given implicitly by (17) and

$$\frac{1}{2H}(\frac{\partial\varphi}{\partial\tau})_w = \frac{V^{1/2}|_{x=\varphi}}{2}\int_1^{\varphi}\frac{dx}{V^{3/2}}\left(x\left(\dot{H} - \frac{k}{a^2}\right) \mp \frac{2\mu}{a^4}\left(1 + \frac{4\alpha\Lambda}{3} + \frac{8\alpha\mu}{a^4x^2}\right)^{-3/2}\right) \tag{19}$$

Note that $\varphi(\tau,0) = 1$ and $\partial_\tau\varphi(\tau,0) = 0$ so $\psi(\tau,0) = 1$. That is the induced metric at $w = 0$ is Robertson-Walker with scale factor $a(\tau)$.

We define $S^2(\tau,w) = a^2(\tau)\varphi(\tau,w)$ and $N^2(\tau,w) = \psi^2(\tau,w)/\varphi(\tau,w)$ and calculate the 00-component of (14) to obtain

$$2\frac{S_0'}{S_0}\left(3 + 12\alpha\left(\frac{\dot{S}_0^2}{N_0^2S_0^2} + \frac{k}{S_0^2}\right) - 4\alpha\frac{{S'}_0^2}{S_0^2}\right) = -\rho \tag{20}$$

where $S_0' = S'(0^+) = -S'(0^-)$ fixed at that value from the product of the normal vector with the extrinsic curvature as explained earlier. Note that $S_0 = S(\tau,0) = a(\tau)$ and $H = \dot{S}_0/S_0 = \dot{a}(\tau)/a(\tau)$ with $N_0 = N(\tau,0) = 1$.

Taking the square of that equation, we see that we only need the quantity ${S'}_0^2/S_0^2$, which is obtained by using equation (17) and the definition of $S(\tau,w)$. We have

$$\begin{aligned}\frac{{S_0'}^2}{S_0^2} &= \frac{1}{4}(\frac{\partial\varphi}{\partial w})^2_{w=0} = V(x=1) = H^2 + \frac{f(a)}{a^2} = \\ &= H^2 + \frac{k}{a^2} + \frac{1}{4\alpha}\left(1 \pm \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{8\alpha\mu}{a^4}}\right)\end{aligned} \tag{21}$$

Substituting in (20) we obtain

$$\begin{aligned}&4(H^2 + \frac{k}{a^2} - \phi)(3 + 8\alpha(H^2 + \frac{k}{a^2}) + 4\alpha\phi)^2 = \rho^2, \\ &\phi = -\frac{1}{4\alpha}\left(1 \pm \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{8\alpha\mu}{a^4}}\right)\end{aligned} \tag{22}$$

The single real solution of the equation above with a smooth limit $\alpha \to 0$ (corresponding to the minus sign choice for the functions $f(y)$ and ϕ), can be written in the form

$$\begin{aligned}&H^2 = -\frac{1}{4\alpha} + \frac{(\phi + \frac{1}{4\alpha})^2}{Q^{2/3}} + \frac{Q^{2/3}}{4} - \frac{k}{a^2}, \\ &Q = \frac{1}{4\alpha}\left(\rho + \sqrt{\rho^2 + 128\alpha^2(\phi + \frac{1}{4\alpha})^3}\right)\end{aligned} \tag{23}$$

The first order corrections to the result of [1] can be read from

$$H^2 + \frac{k}{a^2} = \left(1 - \frac{4\alpha\Lambda}{3} - \frac{72\alpha\mu}{a^4}\right)\frac{\rho^2}{36} - \alpha\frac{\rho^4}{243} + \frac{\Lambda}{6}\left(1 - \frac{\alpha\Lambda}{3}\right) + \frac{\mu(1 - 6\alpha\Lambda)}{a^4} - \alpha\frac{2\mu^2}{a^8} \tag{24}$$

Shifting the energy density, $\rho = \eta + \varrho$, and tuning η so that to have a vanishing 4d cosmological constant, we obtain

$$H^2 + \frac{k}{a^2} = \frac{\sqrt{-\Lambda}}{3\sqrt{6}} \left(1 + \frac{\alpha\Lambda}{2} - \frac{8\alpha\mu}{a^4}\right) \varrho + \frac{\mu(1 + \frac{2}{3}\alpha\Lambda)}{a^4} - \frac{2\alpha\mu^2}{a^8} + \mathcal{O}(\varrho^2) \tag{25}$$

The effective 4d coupling constant has become scale factor dependent. [5]

Now let us see how the same result arises when we treat the 3-wall as a body in the bulk. Calculating the 00-component of the bulk field equations we obtain

$$\frac{3S''}{S} - 12\alpha\frac{S''S'^2}{S^3} + 12\alpha\frac{kS''}{S^3} + 12\alpha\frac{S''\dot{S}^2}{N^2S^3} + .. = -\rho\delta(w) \tag{26}$$

where dots contain terms involving only first w-derivatives and $S' = \partial_w S$ and $\dot{S} = \partial_\tau S$.

We integrate both sides in an infinitesimal region around zero. The first derivative of the metric functions, such as φ, are taken to change sign passing through the point $w = 0$ as is actually implied by (17). With that in mind one can write

$$\partial_w \left(\frac{3S'}{S} - 4\alpha\frac{S'^3}{S^3} + 12\alpha\frac{kS'}{S^3} + 12\alpha\frac{S'\dot{S}^2}{N^2S^3}\right) + .. = -\rho\delta(w) \tag{27}$$

where dots are first derivative terms that contribute zero to the integral. Then

$$\frac{3[S_0']}{S_0} - 4\alpha\frac{[S'^3_0]}{S_0^3} + 12\alpha\frac{k[S_0']}{S_0^3} + 12\alpha\frac{[S_0']\dot{S}_0^2}{N_0^2S_0^3} = -\rho \tag{28}$$

where $[S'] = S'(0^+) - S'(0^-)$. $\mathbf{Z_2}$ symmetry implies that $S'(0^+) = -S'(0^-)$ that is $[S'] = 2S'(0^+) = 2S_0'$ and $S'^2(0^+) = S'^2(0^-) = S'^2_0$. Then, we obtain the same Friedmann equation. This result is agreement with the results of [9] and [16] and the analysis of [13].

In [14] it was argued that the quantities that appear in these formulas make the expression not well defined from the point of view of distributions. This due to the existence of a term involving S'^2S'', as S'^2 is a discontinuous function. On the other hand, this can be combined to $(S'^3)'$ which is well defined and still a delta function, as the derivative of both the sign function and its cube behave as delta functions multiplied with smooth functions. A real difficulty would arise only in the case of product of distributions, as in the loop calculations in quantum field theory, where this leads to an introduction of a cutoff. This is the conclusion of [14] for the problem of the Friedmann equation on the brane in the Gauss-Bonnet theory, where based on that, it is argued that the equation does not change, only the coupling constants, as in the renormalization of loop graphs.

The same is suggested in [15], where finite results have been found, by treating the sign function squared $\epsilon^2(w)$, in products with the delta function, as a constant function at the value one. On the other hand integrating the functions $\epsilon(w)$ and $\epsilon^2(w)$ with the delta function, it is clear that the usual rules are not obeyed when one assigns prescribed values to them at $w = 0$, through irrelevant limiting procedures.

Concluding the main discussion, we would like to emphasize that the procedure of obtaining the junction conditions by the appropriate boundary term makes clear that the domain wall in an Einstein-Gauss-Bonnet theory is a surface of zero thickness as much as it is in the Einstein theory. This is because one is working directly with the surface of the boundary and there is no room for any regularization in the normal direction.

Finally, we briefly discuss theories with additional bulk fields. Assume for example that the gravity includes the dilaton field and its action is given by

$$S = \int d^5x \left(-2\Lambda - \frac{1}{2}(\nabla\Phi)^2 + R + \alpha\, h(\Phi)(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\right) \tag{29}$$

[5]For the plus sign choice in the definition of $f(y)$ the solution (15) is classically unstable [8]. A simple analysis of the equation (22) shows that in this case (where $\phi + \frac{1}{4\alpha} < 0$), the condition for a single real solution is $\rho^2 + 128\alpha^2(\phi + \frac{1}{4\alpha})^3 > 0$. Otherwise there are three real solutions. For $\phi + \frac{1}{4\alpha} > 0$ there is only one real solution, eq. (23), without constraint.

This is just one convenient choice so that to obtain a Friedmann equation with a dilaton type field present. A general solution for the cosmology on the brane in this theory, is equivalent to this 3-surface of constant spatial curvature, moving in a general background in the 5d spacetime with the same property. Assume then, that a certain general static solution of constant spatial curvature, as in (15) with a different $f(y)$, is known, as well as the form of the dilaton $\Phi = \Phi(y)$ [6]. By integrating the 00-component of the field equations, calculated for the line element in the Gaussian normal coordinates form, around $w = 0$ and following steps explained above, we obtain

$$4\left(H^2+\frac{f(a)}{a^2}\right)\left(-3+4\alpha\left(h+3a\frac{dh}{da}\right)\left(H^2+\frac{f(a)}{a^2}-3\left(H^2+\frac{k}{a^2}\right)\right)+24\alpha a\frac{dh}{da}\frac{k}{a^2}\right)^2=\rho^2 \tag{30}$$

where $a = a(\tau)$ as above and $h = h(\Phi(a))$ and we have used the relation $S'^2_0/S_0^2 = H^2 + f(a)/a^2$ as shown above (eq. (21). For $h = 1$ this goes over to equation (22). It is still a cubic equation with respect to the Hubble parameter H^2 but with scale factor dependent coefficients.

For the case of a bulk form-field, practically only the black hole changes so that the Hubble parameter still satisfies a cubic equation similar to (22). Such cases has been studied at least in [18].

Note added: While this work had been at the final stages S. C. Davis reported similar analysis in [16].

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